**Stochastic Process : Final exam 2016**

Time: 3 hours  Maximum Marks 100

Answer any 4 questions. Each question carries 25 marks.

1. [15+10]

(a) Consider an irreducible Markov chain $\{X_n\}$ with state space $S$ with transition probability matrix $(p_{ij})$. Suppose the (common) period of each state is 3. Consider the process $Y_k = X_{3k}$. Show that $\{Y_k\}$ is a Markov chain. Obtain the transition probability matrix $(q_{ij})$ of $\{Y_k\}$ in terms of $(p_{ij})$. Show that $\{Y_k\}$ is aperiodic but is reducible.

(b) Let $\{X_n : n \geq 1\}$ be an $S$-valued Markov Chain with stationary transition probabilities $(p_{ij})$ where $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let $(p_{ij})$ be defined by $p_{11} = 0.7$, $p_{12} = 0.3$, $p_{21} = 0.3$, $p_{28} = 0.7$, $p_{35} = 0.25$, $p_{37} = 0.75$, $p_{44} = 0.75$, $p_{46} = 0.25$, $p_{57} = 0.75$, $p_{58} = 0.25$, $p_{66} = 0.25$, $p_{64} = 0.75$, $p_{75} = 0.2$, $p_{77} = 0.8$, $p_{82} = 0.7$, $p_{88} = 0.3$. Classify the states into transient and recurrent states. For recurrent states $j, k \in S$ write down the value of

$$\lim_{n\to\infty} p_{jk}^{(n)}.$$

2. Let $\{X_n\}$ be a $S$ valued Markov Chain with stationary transition probabilities $(p_{ij})$ and initial distribution $\pi$ which is a stationary distribution *i.e.* it satisfies $\sum_{i\in S} \pi_i p_{ij} = \pi_j$. Recall $f_{ij}^{(m)} = P(A_m \mid X_0 = i)$ where $A_m = \{X_m = j,\ X_t \neq j,\ 1 \leq t < m\}$. Let $\nu_j = \sum_{j=1}^{\infty} n f_{jj}^{(n)}$. Show that [5+5+5+5+5]

(a) $P(A_n, X_0 = j) = f_{jj}^{(n)} \pi_j$.

(b) $P(A_n, X_0 = j) = P(A_n) - P(A_n, X_0 \neq j)$.

(c) $P(A_n, X_0 \neq j) = P(A_{n+1})$.

(d) $\sum_{n=m}^{\infty} f_{jj}^{(n)} \pi_j = P(A_m)$.

(e) $\nu_j \pi_j = 1$.

[You may use without proof the fact that since the initial distribution is stationary, for $V \subset S^m$, $n \geq 1$ $m \geq 1$,

$$P(\ (X_n, X_{n+1}, \ldots, X_{n+m-1}) \in V) = P((X_0, X_1, \ldots, X_{m-1}) \in V).$$

3. Let $\{N_t\ :\ t \geq 0$ be a Poisson process with rate $\lambda$ and let $\{X_n\ :\ n \geq 0\}$ be a Markov chain taking values in $S = \{0, 1, 2, \ldots\}$ with stationary transition probabilities $(p_{ij})$. Assume that $X$ and $N$ are independent. Also suppose $\{X_n\ :\ n \geq 0\}$ admits a stationary distribution $(\pi_i)$. [9+9+7]
Consider the process

$$Z_t = X_{N_t}.$$

(a) Show that for $0 \leq s_0 \leq s_1 \leq \ldots \leq s_k \leq s \leq s+t$, $i_0, i_1, \ldots, i_k, i, j \in S$

$$\begin{aligned} P(Z_{t+s} = j, N_{t+s} = m+n \mid Z_{s_0} = i_0, N_{s_0} = n_0, \ldots, Z_{s_k} = i_k, N_{s_k} = n_k, Z_s = i, N_s = n) \\ = P(X_m = j \mid X_0 = i) P(N_t = m) \\ = \sum_{m=0}^{\infty} p_{ij}^{(m)} \exp(-\lambda t) \frac{(\lambda t)^m}{m!}. \end{aligned}$$

(b) Show that $\{Z_t\}$ is a Markov Chain with Stationary transition probabilities

$$P(Z_{t+s} = j \mid Z_{s_0} = i_0, \ldots, Z_{s_k} = i_k, Z_s = i) = P_{ij}(t)$$

where $0 \leq s_0 \leq s_1 \leq \ldots \leq s_k \leq s \leq s+t$, $i_0, i_1, \ldots, i_k, i, j \in S$ and obtain an expression for $P_{ij}(t)$ in terms of $(p_{ij})$ and $\lambda$.

(c) Show that and for $i \neq j$

$$\frac{d}{dt}P_{ij}(t)\mid_{t=0} = \lambda p_{ij}, \quad \frac{d}{dt}P_{ii}(t)\mid_{t=0} = -\lambda(1-p_{ii}).$$

4. [12+13]

(a) Let S={0,1,2,...} and let $(p_{ij})$ be defined as follows: Let $0 < p < 1$, $0 < \theta < 1$ be fixed and let $q = 1-p$. Let $p_{00} = \theta$, $p_{01} = 1-\theta$ and for $i \geq 1$ let $p_{i,i-1} = q$ and $p_{i,i+1} = p$. (for all other $i, j$, $p_{ij} = 0$). Show that the chain is irreducible aperiodic. For what values of $p, \theta$ does the chain admit stationary initial distribution? Obtain the stationary initial distribution when it exists.

(b) Let $S = \{0, 1, 2, \ldots, \}$ and let $\{X_n : n \geq 1\}$ be an $S$-valued Markov Chain with stationary transition probabilities $(p_{ij})$ and initial distribution $\{\mu_i = P(X_0 = i),\ i \in S\}$. Suppose

$$p_{01} = 1,\ p_{i,i+1} = a_i,\ p_{i,0} = 1 - a_i, \text{ for all } i \geq 1$$

where $0 < a_i < 1$ for all $i \geq 1$ with all other transition probabilities being zero. Show that the Markov chain $\{X_n\}$ is irreducible and aperiodic . Find conditions on $\{a_i\}$ under which the chain is (i) transient, (ii) recurrent (iii) positive recurrent and (iv) null recurrent.

Recall: A recurrent chain is positive recurrent if and only if it admits a stationary initial distribution and other wise it is null recurrent.

5. Recall that for finite sets $E, F$, where $E$ is subset of real numbers, for random variables $X, Z$, such that $X \in E$ and $Z \in F$, the conditional expectation $\mathbb{E}[X \mid Z]$ of $X$ given $Z$, is defined as follows:

$$\mathbb{E}(X \mid Z) = g(Z)$$

where $g : F \mapsto \mathbb{R}$ is defined by, for $\gamma \in F$ such that $\mathbb{P}(Z = \gamma) > 0$, writing $B_\gamma = \{\omega : Z(\omega) = \gamma\}$

$$g(\gamma) = \frac{1}{\mathbb{P}(B_\gamma)} \sum_\omega \mathbf{I}_{B_\gamma}(\omega) X(\omega) P(\{\omega\})$$

and for $\gamma \in F$ such that $\mathbb{P}(Z = \gamma) = 0$, $g(\gamma) = 0$. [5+8+6+6]

(a) Show that $\mathbb{E}[g(Z)] = \mathbb{E}[X]$.

(b) For a function $u$ on $F$ show that $\mathbb{E}[u(Z)X \mid Z] = u(Z)\mathbb{E}[X \mid Z]$.

(c) Show that for any function $h : F \mapsto \mathbb{R}$ $\mathbb{E}[(X - g(Z))h(Z)] = 0$.

(d) For all functions $h$ on $F$ $\mathbb{E}[(X - g(Z))^2] \leq \mathbb{E}[(X - h(Z))^2]$.

6. Let us fix a sequence of random variables $\{S_0, S_1, \ldots, S_N\}$ such that each $S_n$ takes values in a finite set $E$ and $S_0$ is a constant. All notions, adapted, predictable, stop time martingale, refers to $S_0, S_1, \ldots S_k$ as observable at time $k$. Let $\{M_n : n \geq 1\}$ be a martingale (with $M_0 = \mathbb{E}(M_1)$), $\tau, \sigma$ be stopping times with $0 \leq \sigma \leq \tau \leq N$, For $1 \leq n \leq N$ let, $A_n = \mathbf{1}_{\{\sigma < k \leq \tau\}}$, $Y_n = M_n^2$, $V_n = \sum_{k=1}^n A_k(Y_k - Y_{k-1})$. Show that [5+5+5+5+5]

(a) $\{Y_n : n \geq 1\}$ is a sub-martingale.

(b) $\{A_n : n \geq 1\}$ is predictable.

(c) $\{V_n : n \geq 1\}$ is a sub-martingale.

(d) $V_n = Y_{\tau \wedge n} - Y_{\sigma \wedge n}$.

(e) $\mathbb{E}(M_\sigma^2) \leq \mathbb{E}(M_\tau^2)$.

[You may use any part of any question while solving any other part of the same or different question, even if you do not solve the part you are invoking- just refer to as part (x) of question no (y).]